

A new proof of a characterization of small spherical caps

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Abstract

It is known that planar disks and small spherical caps are the only constant mean curvature graphs whose boundary is a round circle. Usually, the proof invokes the Maximum Principle for elliptic equations. This paper presents a new proof of this result motivated by an article due to Reilly. Our proof utilizes a flux formula for surfaces with constant mean curvature together with integral equalities on the surface.

1 Introduction and the result

A surface in Euclidean space \mathbb{R}^3 with the property that its mean curvature is constant at each point is called a *constant mean curvature surface* or CMC surface for short. Round spheres are closed CMC surfaces. Here by closed surface we mean compact and without boundary surface. A famous theorem due to Hopf asserts that any closed CMC surface of genus 0 must be a round sphere [4]. Later, Alexandrov proved in 1956 that any embedded closed CMC surface in \mathbb{R}^3 must be a round sphere [1]. For a long time it was an open question whether or not spheres were the only closed CMC surfaces in \mathbb{R}^3 . If a such surface were to exist, it would necessarily be a surface

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with self-intersections and higher genus. In 1986, Wente found an immersed torus with constant mean curvature [14]. This discovery inspired a great deal of work in the search for new examples of closed CMC surfaces. For readers interested in the subject, we refer the recent survey [6] and references therein.

We now consider compact CMC surfaces with non-empty boundary. The simplest case for the boundary is a round circle. If C is a circle of radius $r > 0$, we consider C in a sphere $S(R)$ of radius R , $R \geq r$. The mean curvature of $S(R)$ is $H = 1/R$ with respect to the inward orientation. Then C splits $S(R)$ in two spherical caps with the same boundary C and constant mean curvature H . If $R = r$, both caps are hemispheres whereas if $R > r$, there are two geometrically distinct caps which we call the small and the big spherical cap. On the other hand, the planar disk bounded by C is a compact surface with constant mean curvature $H = 0$. These surfaces are the only totally umbilic compact CMC surfaces bounded by C .

In 1991, Kapouleas found other examples of CMC surfaces bounded by a circle [5]. The surfaces that he obtained have higher genus and self-intersections. Thus, one asks under what conditions a compact CMC surface bounded by a circle is spherical. Taking into account the theorems of Hopf and Alexandrov for closed surfaces above cited, the natural hypotheses to consider for surfaces bounded by a circle is that either S has the simplest possible topology, that is, the topology of a disk, or that S is embedded. Surprisingly, we have

Conjeture 1. Planar disks and spherical caps are the only compact CMC surfaces bounded by a circle that are *topological disks*.

Conjeture 2. Planar disks and spherical caps are the only compact CMC surfaces bounded by a circle that are *embedded*.

This means that our knowledge about the structure of the space of CMC surfaces bounded by a circle is quite limited and only several partial results have been obtained by different authors (we refer to [6] again). Of course, the methods of proof for the Hopf and Alexandrov Theorems can not be applied with complete success in the context of a non-empty boundary. This fact, together the lack of examples, suggests that although the problems in the non-empty boundary case have the same flavor as in the closed one, the proofs are more difficult.

A partial answer to the conjecture 2 is the following

Theorem 1 (Alexandrov) *Let C be a round circle in a plane Π and let S be an embedded compact CMC surface bounded by C . If S lies in one side of Π , then S is a planar disk or a spherical cap.*

The extra hypothesis that we add is that S lies on one side of the plane containing the boundary. Although Alexandrov did not state this result, the proof is accomplished using the same technique that he used in proving his theorem that was stated above: the so-called Alexandrov reflection method. Behind this method lies the classical Maximum Principle for elliptic partial differential equations, together with the moving plane technique. A particular case of this Theorem is the following. Given H and a circle C , among the two spherical caps bounded by C with mean curvature H , only the small one is a graph. Using this method, we characterize the small spherical caps as

Theorem 2 *Let C be a round circle in a plane Π and let S be a compact CMC surface bounded by C . If S is a graph over Π , then S is a planar disk or a small spherical cap.*

The purpose of this article is to give a new proof of this result and that *does not involve the Maximum Principle*. This different approach, that is, avoiding the Maximum Principle, appeared in the closed case which motivated the present work. In 1978, Reilly obtained another proof of the Alexandrov theorem for CMC closed surfaces without the use of the Maximum Principle thanks to a combination of the Minkowski formulae with some new elegant arguments [12]. In this sense, a new elementary proof of Alexandrov's Theorem due to Ros appears in [13].

In the same spirit, we will use integral formulae together a type of "flux formula". Moreover we will see in the next section how the equation that characterizes a CMC surface can be expressed in terms of the Laplace operator. This was already noticed by Reilly as one can see from the title of his article. This fact allows us to establish some geometrical properties about CMC surfaces using basic multivariable Calculus.

2 Mean curvature, graphs and the Laplacian

Let S be a surface in \mathbb{R}^3 and which we write locally as the graph of a smooth function f , $z = f(x, y)$, (x, y, z) being the usual coordinates of \mathbb{R}^3 . We orient S with the choice of normal given by

$$N(x, y, f(x, y)) = \frac{(-f_x, -f_y, 1)}{\sqrt{1 + f_x^2 + f_y^2}}(x, y), \quad (1)$$

where the subscripts indicate the corresponding partial derivatives. The mean curvature H of S satisfies the following partial differential equation:

$$2H(1 + f_x^2 + f_y^2)^{\frac{3}{2}} = (1 + f_y^2)f_{xx} - 2f_x f_y f_{xy} + (1 + f_x^2)f_{yy}. \quad (2)$$

Equation (2) may be written as

$$\operatorname{div} \left(\frac{\nabla f}{\sqrt{1 + |\nabla f|^2}} \right) = 2H, \quad (3)$$

where div and ∇ stand for the divergence and gradient operators respectively. In PDE theory, this equation falls into the category of elliptic type, whose main property is the existence of a Maximum Principle: if two functions f_1 and f_2 satisfy equation (3) with the same Dirichlet condition, then $f_1 = f_2$. We refer the reader to [3, sect. 10.5]. For CMC surfaces, this geometrically translates to the assertion that if two CMC surfaces with the same constant mean curvature intersect tangentially at some point and one surface lies locally on one side of the other, then both surfaces must coincide in a neighborhood of that point. This property was used by Alexandrov in proving his theorem by using a surface as a comparison surface with itself in a reflection process. When the surface has non-empty boundary, we must add the extra hypothesis that S lies over Π as states Theorem 1. By doing this, we avoid the presence of a possible contact between an interior point with a boundary point of the surface where the Maximum Principle fails. We refer the reader to [6, 7] for detailed proofs of Theorem 1.

We prove two results about CMC surfaces which have their own geometric interest (they will not be used later). Both results are well known in the literature although usually the Maximum Principle is invoked in the proofs. However, we show them by using a basic knowledge of calculus and differential geometry.

Theorem 3 *Let D be a domain of a plane Π and let S be a compact CMC graph on D whose boundary is ∂D . If $H \neq 0$, then $\operatorname{int}(S)$ lies in one side of Π .*

Proof: We argue by contradiction. Assume that S has (interior) points on both sides of Π . Consider $p_0 = (x_0, y_0, z_0)$, $z_0 > 0$ and $p_1 = (x_1, y_1, z_1)$, $z_1 \leq 0$, points of S with highest and lowest height z , respectively. If S is the graph of a function $z = f(x, y)$, then,

$$f_x(x_i, y_i) = f_y(x_i, y_i) = 0, \quad i = 0, 1 \quad (4)$$

$$f_{xx}(x_0, y_0), f_{yy}(x_0, y_0) \leq 0, \quad f_{xx}(x_1, y_1), f_{yy}(x_1, y_1) \geq 0. \quad (5)$$

Using the orientation given by (1), let us compute the mean curvature H at p_0 and p_1 . Because H is constant and using (4) and (5), equation (2) leads to

$$2H = 2H(p_0) = (f_{xx} + f_{yy})(x_0, y_0) \leq 0 \leq (f_{xx} + f_{yy})(x_1, y_1) = 2H(p_1) = 2H. \quad (6)$$

Since $H \neq 0$, we get a contradiction.

q.e.d

The inequalities in (6) can be written as

$$2H(p_0) = \Delta_0 f(x_0, y_0) \leq 0 \leq \Delta_0 f(x_1, y_1) = 2H(p_1),$$

where $\Delta_0 = \partial_{xx} + \partial_{yy}$ is the Euclidean Laplacian. This indicates that under a certain choice of coordinates, (2) can be expressed in terms of the Laplace operator. See also [12].

We treat the minimal case, that is, $H = 0$.

Theorem 4 *Consider a Jordan curve C in a plane Π . If S is a compact CMC surface with $H \equiv 0$, whose boundary is C , then S is the planar domain D that bounds C .*

Proof: We point out that we have dropped the hypothesis that S is a graph. We use a similar proof as in Theorem 3 and we follow the notation used there. The reasoning is by contradiction again. Without loss of generality, we assume that S has points over Π . Let $\Gamma \subset \Pi$ be a circle of radius r sufficiently large so that D lies strictly inside of the circular disk determined by Γ and so that the hemisphere K with $\partial K = \Gamma$ over Π also lies over S . Let $S(H)$ be the family of small spherical caps over Π with $\partial S(H) = \Gamma$ and parameterized by their mean curvature H oriented by (1). Then $-1/r < H < 0$. In the limit case, $S(-1/r) = K$. Beginning from the value $H = -1/r$, we let $H \rightarrow 0$ until the first value of h , $-1/r < h < 0$, that $S(h)$ touches the original surface S . Let p_0 be the contact point. Both surfaces are locally graphs of functions defined in the (common) tangent plane at p_0 . This point is not necessarily the highest point of S , but we do a change of coordinates so that p_0 is the origin, the tangent plane of $S(h)$ and S at p_0 is the xy -plane and $S(h)$ lies over S in a neighborhood of p_0 . Now p_0 is the highest point of S and both surfaces lie below Π .

Consider the two functions f and g whose graphs are S and $S(h)$ respectively and defined in some planar domain Ω of Π containing the origin. Let $u = f - g$. Then $u \leq 0$ on Ω with a local maximum at $(0, 0)$. Because $g > 0$ on ∂D , p_0 is an interior point of Ω . Consequently, $f_x(0, 0) = f_y(0, 0) = g_x(0, 0) = g_y(0, 0) = 0$ and

$$u_{xx}(0, 0), \quad u_{yy}(0, 0) \leq 0. \tag{7}$$

However at the point p_0 , equation (2) for S and $S(h)$ is

$$0 = (f_{xx} + f_{yy})(0, 0) \quad \text{and} \quad -\frac{1}{r} = (g_{xx} + g_{yy})(0, 0),$$

respectively. By subtracting both equations, we obtain $1/r = (u_{xx} + u_{yy})(0, 0) > 0$, contradicting (7).

q.e.d

3 The effect of the boundary in the shape of a CMC surface

We have seen that if C is a circle of radius r , the possible values of the mean curvatures H for spherical caps bounded by C lies in the range $[-1/r, 1/r]$ because $R \geq r$ for the radius of the spheres $S(R)$. Thus, the boundary C imposes restrictions on the possible values of mean curvature. We show that this occurs for a general curved boundary. Consider a compact CMC surface S with boundary $\partial S = C$ and let Y be a variation field in \mathbb{R}^3 . The first variation formula of the area $|A|$ of the surface S along Y is

$$\delta_Y |A| = -2H \int_S \langle N, Y \rangle \, dS - \int_{\partial S} \langle \nu, Y \rangle \, ds,$$

where N is the unit normal vector of S , H is the mean curvature relative to N , ν represents the inward unit vector along ∂S and ds is the arc-length element of ∂S . Let us fix a vector $\vec{a} \in \mathbb{R}^3$ and consider Y as the generating field of a family of translations in the direction of \vec{a} . As Y generates isometries of \mathbb{R}^3 , the first variation of A vanishes and thus

$$2H \int_S \langle N, \vec{a} \rangle \, dS + \int_{\partial S} \langle \nu, \vec{a} \rangle \, ds = 0. \quad (8)$$

The first integral transforms into an integral over the boundary as follows. The divergence of the field $Z_p = (p \times \vec{a}) \times N$, $p \in S$, is $-2\langle N, \vec{a} \rangle$ (here \times denotes the cross product of \mathbb{R}^3). The Divergence Theorem, together with (8), yields

$$-H \int_{\partial S} \langle \alpha \times \alpha', \vec{a} \rangle \, ds = \int_{\partial S} \langle \nu, \vec{a} \rangle \, ds, \quad (9)$$

where α is a parametrization of ∂S such that $\alpha' \times \nu = N$.

This equation known as the “balancing formula” or “flux formula” is due to R. Kusner (see [9]; also in [10, 8]). It is a conservation law in the sense of Noether that reflects the fact that the area (the potential) is invariant under the group of translations of Euclidean space. On the other hand, if D is a 2-cycle with boundary ∂S , the formula can be viewed as expressing the physical equilibrium between the

force of exterior pressure acting on D (the left-hand side in (9)) with the force of surface tension of S that act along its boundary (the right-hand side).

If the boundary C lies in the plane $\Pi = \{x \in \mathbb{R}^3; \langle x, \vec{a} \rangle = 0\}$, for $|\vec{a}| = 1$, then (9) gives

$$2H\bar{A} = \int_{\partial S} \langle \nu, \vec{a} \rangle \, ds, \quad (10)$$

where \bar{A} is the algebraic area of C . Given a closed curve $C \subset \mathbb{R}^3$ that bounds a domain D , and noting that $\langle \nu, \vec{a} \rangle \leq 1$, the possible values of the mean curvature H of S satisfy

$$|H| \leq \frac{\text{length}(C)}{2 \, \text{area}(D)}. \quad (11)$$

In particular, if C is a circle of radius $r > 0$, a necessary condition for the existence of a surface spanning C with constant mean curvature H is that $|H| \leq 1/r$.

Remark 1 If S is the graph of $z = f(x, y)$ it follows from the Divergence Theorem and (3) that

$$\begin{aligned} 2|H|\text{area}(D) &= \left| \int_D 2H \, dD \right| = \left| \int_{\partial D} \left\langle \frac{\nabla f}{\sqrt{1 + |\nabla f|^2}}, \vec{n} \right\rangle \, ds \right| \\ &\leq \int_{\partial D} \frac{|\nabla f|}{\sqrt{1 + |\nabla f|^2}} \, ds < \int_{\partial D} 1 \, ds = \text{length}(C), \end{aligned}$$

where \vec{n} is the unit normal vector to ∂D in Π . Then,

$$|H| < \frac{\text{length}(C)}{2 \, \text{area}(D)}.$$

As consequence of Remark 1, the proof of Theorem 2 is very simple by using the Maximum Principle as we show at this time. If S is the graph of a function $z = f_1(x, y)$ and H is its mean curvature, then $|H| < 1/r$, where r is the radius of C . But there exists a small spherical cap with the same boundary and mean curvature as S . As this cap is the graph of a function f_2 , we have that f_1 and f_2 are two solutions of (3) with the same Dirichlet condition, the Maximum Principle implies $f_1 = f_2$. Other proof of Theorem 2 using a combination of the flux formula and the Maximum Principle appears in [2].

4 A new proof of Theorem 2

In this section we will prove our result without the use of the Maximum Principle. Let S satisfy the hypotheses of Theorem 2 and let (x, y, z) be the usual coordinates of \mathbb{R}^3 . Without loss of generality, we assume that Π is the xy -plane, that is, $\Pi = \{z = 0\}$ and that C is a circle of radius $r > 0$ centered at the origin. Let $\vec{a} = (0, 0, 1)$.

Consider the unit normal vector N given by (1). Then $\langle N, \vec{a} \rangle > 0$ on S . We will use the notation of Section 3. First, let α be the parametrization of C such that $\alpha' \times \nu = N$. We know that $\alpha'' = -\alpha/r^2$ (independent of the orientation of C) and since $\langle N, \vec{a} \rangle > 0$ along ∂S , we have $\alpha \times \alpha' = r\vec{a}$. Then

$$\langle \nu, \vec{a} \rangle = \langle N \times \alpha', \vec{a} \rangle = \langle N, \alpha' \times \vec{a} \rangle = \frac{1}{r} \langle N, \alpha \rangle. \quad (12)$$

The integral equation (9) gives

$$-2\pi r^2 H = \int_C \langle \nu, \vec{a} \rangle \, ds. \quad (13)$$

and thus, equation (8) is

$$\int_S \langle N, \vec{a} \rangle \, dS = \pi r^2. \quad (14)$$

We will need the following result:

Lemma 1 *The function $\langle N, \vec{a} \rangle$ satisfies*

$$\Delta \langle N, \vec{a} \rangle + |\sigma|^2 \langle N, \vec{a} \rangle = 0, \quad (15)$$

where Δ is the Laplace-Beltrami operator on S and σ is the second fundamental form.

Proof: Formula (15) holds for any CMC surface. Let $\mathbf{x} : S \rightarrow \mathbb{R}^3$ be an immersion of a surface in \mathbb{R}^3 . For any vector field Y of the ambient space \mathbb{R}^3 , we consider the decomposition $Y = V + uN$, where V is a tangent vector field to \mathbf{x} and $u = \langle N, Y \rangle$ is the normal component of Y . We consider a smooth variation (\mathbf{x}_t) of \mathbf{x} ($\mathbf{x}_0 = \mathbf{x}$) whose variation vector field is uN , that is, $\partial_t(\mathbf{x}_t)_{t=0} = uN$. Then the variation of the mean curvature H_t of the (\mathbf{x}_t) changes according to

$$\partial_t(H_t)_{t=0} = \frac{1}{2}(\Delta u + |\sigma|^2 u) + \langle \nabla H, V \rangle.$$

The first summand in the above equation is the linearization of the mean curvature operator. See [11].

Assume now that \mathbf{x} is a CMC surface. Then $\nabla H = 0$. Let us take the vector field $Y = \vec{a}$ whose associated one-parameter subgroup generates translations. Thus the mean curvature is fixed pointwise throughout the variation, and so, $\partial_t(H_t)_{t=0} = 0$, proving (15).

q.e.d

We follow with the proof of Theorem 2. By applying the Divergence Theorem to the vector field $\nabla \langle N, \vec{a} \rangle$ and using equation (15), we get

$$\int_S |\sigma|^2 \langle N, \vec{a} \rangle \, dS = \int_C \langle dN\nu, \vec{a} \rangle \, ds. \quad (16)$$

We study each side of (16) beginning with the left-hand side. The inequality $|\sigma|^2 \geq 2H^2$ holds on any surface, and equality occurs only at umbilic points. By using (14) and that $\langle N, \vec{a} \rangle > 0$ on S , the left-hand side of (16) yields

$$\int_S |\sigma|^2 \langle N, \vec{a} \rangle \, dS \geq 2H^2 \int_S \langle N, \vec{a} \rangle \, dS = 2\pi r^2 H^2. \quad (17)$$

We now turn our attention to the right-hand side of (16). First, note

$$dN\nu = -\sigma(\alpha', \nu)\alpha' - \sigma(\nu, \nu)\nu.$$

From (12), we have

$$\begin{aligned} \sigma(\nu, \nu) &= 2H - \sigma(\alpha', \alpha') = 2H + \langle dN\alpha', \alpha' \rangle \\ &= 2H - \langle N, \alpha'' \rangle = 2H + \frac{1}{r^2} \langle N, \alpha \rangle = 2H + \frac{1}{r} \langle \nu, \vec{a} \rangle. \end{aligned} \quad (18)$$

Because $\langle \alpha', \vec{a} \rangle = 0$, and using (13) and (18), we have

$$\begin{aligned} \int_C \langle dN\nu, \vec{a} \rangle \, ds &= - \int_C \sigma(\nu, \nu) \langle \nu, \vec{a} \rangle \, ds = - \int_C \left(2H + \frac{1}{r} \langle \nu, \vec{a} \rangle \right) \langle \nu, \vec{a} \rangle \, ds \\ &= 4\pi r^2 H^2 - \frac{1}{r} \int_C \langle \nu, \vec{a} \rangle^2 \, ds. \end{aligned} \quad (19)$$

We use (13) and the Cauchy-Schwarz inequality as follows:

$$\int_C \langle \nu, \vec{a} \rangle^2 \, ds \geq \frac{1}{2\pi r} \left(\int_C \langle \nu, \vec{a} \rangle \, ds \right)^2 = 2\pi r^3 H^2. \quad (20)$$

Then equation (19) and inequality (20) imply

$$\int_C \langle dN\nu, \vec{a} \rangle ds \leq 2\pi r^2 H^2. \quad (21)$$

Finally, by combining (16), (17) and (21), we obtain

$$2\pi r^2 H^2 \leq \int_S |\sigma|^2 \langle N, \vec{a} \rangle dS = \int_C \langle dN\nu, \vec{a} \rangle ds \leq 2\pi r^2 H^2.$$

Therefore, we have equalities in all the above inequalities. In particular, $|\sigma|^2 = 2H^2$ on S . This means that S is a totally umbilic surface of \mathbb{R}^3 and so, it is an open of a plane or a sphere. Because, the boundary of S is a circle, S is a planar disk or it is a spherical cap. In the latter case, S must be the small spherical cap since S is a graph. This concludes the proof of the theorem.

We point out that the hypothesis that S is a graph has been only used in inequality (17) to assert $|\sigma|^2 \langle N, \vec{a} \rangle \geq 2H^2 \langle N, \vec{a} \rangle$. However, the rest of the proof is valid for any CMC compact surface bounded by a round circle. The reader may then try to use the ideas that underlie the proof of Theorem 2 to derive other results. As an example, we show the following theorem where we replace the hypothesis that the surface is a graph by the hypothesis of non negativity of the Gauss curvature. Again, we follow the spirit of our work and we do not invoke the Maximum Principle.

Theorem 5 *Let S be a compact CMC surface bounded by a round circle C . If the Gauss curvature K is non-negative, then S is a planar disk or a spherical cap.*

Proof: As $\langle N, \vec{a} \rangle \leq 1$ holds, we have $K \langle N, \vec{a} \rangle \leq K$ independent on the sign of $\langle N, \vec{a} \rangle$. Thus, $K \langle N, \vec{a} \rangle \leq H^2$. From (14) and (17) and since $|\sigma|^2 = 4H^2 - 2K$, we have

$$\begin{aligned} \int_S |\sigma|^2 \langle N, \vec{a} \rangle dS &= 4H^2 \int_S \langle N, \vec{a} \rangle dS - 2 \int_S K \langle N, \vec{a} \rangle dS \\ &\geq 2H^2 \int_S \langle N, \vec{a} \rangle dS = 2\pi r^2 H^2. \end{aligned}$$

The proof then follows the same steps as in the proof of Theorem 2 and we conclude that S is umbilic, and hence a planar disk or a spherical cap. *q.e.d*

We end with a comment. It would be interesting to have a proof of Theorem 1, that is, the boundary version of the Alexandrov theorem, without invoking the Maximum Principle for elliptic equations, as was done by Reilly in [12] for the closed case. We do not know a way of applying our arguments to this question.

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